Lecture 8

Factored Forms

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The Concept

The Concept

ODE system from spatial discretization of PDE:

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = A\vec{u} - \vec{f}$$

Consider some splitting of A into $A_1 + A_2$:

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = [A_1 + A_2]\vec{u} - \vec{f}$$

Apply the explicit Euler time-marching method:

$$\vec{u}_{n+1} = [\ I\ \ + hA_1 + hA_2]\vec{u}_n - h\vec{f} + O(h^2)$$

The Concept

Equivalent to

$$\vec{u}_{n+1} = \big[[\ I\ \ + hA_1][\ I\ \ + hA_2] - h^2 A_1 A_2\big]\vec{u}_n - h\vec{f} + O(h^2)$$

Drop h^2 term:

$$\vec{u}_{n+1} = [\ I\ \ + hA_1][\ I\ \ + hA_2]\vec{u}_n - h\vec{f} + O(h^2)$$

Same formal order of accuracy but potentially different stability properties as well as implications for efficiency

Not important with explicit Euler time marching but can be exploited with implicit methods, as we will see

Factoring Space Matrix Operators in 2D

Mesh Indexing Convention

Consider discretization of 2D diffusion equation on a small mesh

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\begin{array}{ccccccc}
 & & \odot & \odot & \odot & \odot & \\
M_y & \odot & 13 & 23 & 33 & 43 & \odot \\
k & \odot & 12 & 22 & 32 & 42 & \odot \\
1 & \odot & 11 & 21 & 31 & 41 & \odot \\
 & & \odot & \odot & \odot & \odot & \\
 & & 1 & j & \cdots & M_x &
\end{array}$$

Mesh indexing in 2D

Data-Base Permutations

$U^{(x)}$ is the vector of unknowns with j running first, then k

$U^{(y)}$ is the vector of unknowns with k running first, then j

$U^{(x)}$ and $U^{(y)}$ are related by a permutation matrix P_{xy}

$$U^{(x)} = P_{xy} U^{(y)} \qquad \text{and} \qquad U^{(y)} = P_{yx} U^{(x)}$$

$$P_{yx} = P_{xy}^{\mathrm{T}} = P_{xy}^{-1}$$

Semi-discrete form can be written in terms of either $U^{(x)}$ or $U^{(y)}$

$$\frac{\mathrm{d}U}{\mathrm{d}t} = A_{x+y} U + (\vec{bc})$$

$$A_{x+y}^{(x)} = P_{xy} \cdot A_{x+y}^{(y)} \cdot P_{yx}$$

Space Splitting and Factoring

In 2D the matrix operator A_{x+y} includes contributions from the approximations of both $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$

$$A_{x+y}^{(x)} = A_x^{(x)} + A_y^{(x)}$$

$$A_{x+y}^{(y)} = A_x^{(y)} + A_y^{(y)}$$

The individual matrices are also related by

$$A_y^{(x)} = P_{xy} A_y^{(y)} P_{yx}$$

$$A_x^{(x)} = P_{xy} A_x^{(y)} P_{yx}$$

Let's consider using second-order centered differences for both

Space Splitting and Factoring

$$A^{(x)}_{x+y} \cdot U^{(x)} = \left[\begin{array}{cccccccccccccc}
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\end{array}\right] . \quad \begin{array}{c}
11 \\ 21 \\ 31 \\ 41 \\ -- \\ 12 \\ 22 \\ 32 \\ 42 \\ -- \\ 13 \\ 23 \\ 33 \\ 43
\end{array}$$

Space Splitting and Factoring

$$
A^{(y)}_{x+y} \cdot U^{(y)} = \left[\begin{array}{ccc|ccc|ccc|ccc}
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\end{array}\right] \cdot \begin{array}{c}
11 \\ 12 \\ 13 \\ -- \\ 21 \\ 22 \\ 23 \\ -- \\ 31 \\ 32 \\ 33 \\ -- \\ 41 \\ 42 \\ 43
\end{array}
$$

Space Splitting and Factoring

The splitting of $A^{(x)}_{x+y}$

$$
A^{(x)}_{x} \cdot U^{(x)} = \left[\begin{array}{cccccccccccccc}
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& & & & | & & & & & | & & x & x & x \\
& & & & | & & & & & | & & & x & x
\end{array}\right] \cdot U^{(x)}
$$

Space Splitting and Factoring

The splitting of $A_{x+y}^{(x)}$

$$A_y^{(x)} \cdot U^{(x)} = \left[\begin{array}{cccccccccccccc}
o & & & & | & o & & & & | & & & & \\
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o & & & & | & o & & & & | & o & & & \\
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 & & & & | & & & & o & | & & & & o
\end{array}\right] \cdot U^{(x)}$$

Space Splitting and Factoring

The splitting of $A_{x+y}^{(y)}$

$$
A_x^{(y)} \cdot U^{(y)} = \left[\begin{array}{ccc|ccc|ccc|ccc}
x & & & x & & & & & & & & \\
 & x & & & x & & & & & & & \\
 & & x & & & x & & & & & & \\[1em]
x & & & x & & & x & & & & & \\
 & x & & & x & & & x & & & & \\
 & & x & & & x & & & x & & & \\[1em]
 & & & x & & & x & & & x & & \\
 & & & & x & & & x & & & x & \\
 & & & & & x & & & x & & & x \\[1em]
 & & & & & & x & & & x & & \\
 & & & & & & & x & & & x & \\
 & & & & & & & & x & & & x
\end{array}\right] \cdot U^{(y)}
$$

Space Splitting and Factoring

The splitting of $A_{x+y}^{(y)}$

$$A_y^{(y)} \cdot U^{(y)} = \left[\begin{array}{ccc|ccc|ccc|ccc}
o & o & & & & & & & & & & \\
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 & & & & & & & & & o & o & \\
 & & & & & & & & & o & o & o \\
 & & & & & & & & & & o & o
\end{array}\right] \cdot U^{(y)}$$

Space Splitting and Factoring

Apply implicit Euler time marching

$$U_{n+1}^{(x)} = U_n^{(x)} + h\Big[A_x^{(x)} + A_y^{(x)}\Big]U_{n+1}^{(x)} + h(\vec{bc})$$

Move $U_{n+1}^{(x)}$ terms to left-hand side

$$\Big[I - hA_x^{(x)} - hA_y^{(x)}\Big]U_{n+1}^{(x)} = U_n^{(x)} + h(\vec{bc}) + O(h^2)$$

Factor and neglect $O(h^2)$ factorization error

$$\Big[I - hA_x^{(x)}\Big]\Big[I - hA_y^{(x)}\Big]U_{n+1}^{(x)} = U_n^{(x)} + h(\vec{bc}) + O(h^2)$$

Space Splitting and Factoring

$$\left[I - hA_x^{(x)}\right]\left[I - hA_y^{(x)}\right]U_{n+1}^{(x)} = U_n^{(x)} + h(\vec{bc}) + O(h^2)$$

Two linear systems must be solved – split into separate lines

$$\begin{aligned} \left[I - hA_x^{(x)}\right]\tilde{U}^{(x)} &= U_n^{(x)} + h(\vec{bc}) \\ \left[I - hA_y^{(y)}\right]U_{n+1}^{(y)} &= \tilde{U}^{(y)} \end{aligned}$$

In each case, use the database that leads to a tridiagonal matrix

Problem has been reduced to the solution of tridiagonal matrices, which are much faster to solve than the original matrix A_{x+y}

First-order accuracy maintained – what about stability, convergence, and accuracy of steady-state solution?

Second-Order Factored Implicit Methods

Second-Order Factored Implicit Methods

Can we maintain second-order accuracy with a factored form?

Let's check the trapezoidal method

$$\left[I - \frac{1}{2}hA_x - \frac{1}{2}hA_y\right]U_{n+1} = \left[I + \frac{1}{2}hA_x + \frac{1}{2}hA_y\right]U_n + h(\vec{bc}) + O(h^3)$$

Factor both sides

$$\left[\left[I - \frac{1}{2}hA_x\right]\left[I - \frac{1}{2}hA_y\right] - \frac{1}{4}h^2 A_x A_y\right]U_{n+1}$$

$$= \left[\left[I + \frac{1}{2}hA_x\right]\left[I + \frac{1}{2}hA_y\right] - \frac{1}{4}h^2 A_x A_y\right]U_n + h(\vec{bc}) + O(h^3)$$

Terms we wish to neglect are $O(h^3)$ because $U_{n+1} - U_n = O(h)$

Second-Order Factored Implicit Methods

Factored form of trapezoidal method retains second-order accuracy

$$\begin{aligned}\left[I - \frac{1}{2}hA_x\right]\left[I - \frac{1}{2}hA_y\right]U_{n+1} &= \left[I + \frac{1}{2}hA_x\right]\left[I + \frac{1}{2}hA_y\right]U_n \\ &\quad + h(\vec{bc}) + O(h^3)\end{aligned}$$

$$\begin{aligned}\left[I - \frac{1}{2}hA_x\right]\tilde{U} &= \left[I + \frac{1}{2}hA_y\right]U_n + \frac{1}{2}hF_n \\ \left[I - \frac{1}{2}hA_y\right]U_{n+1} &= \left[I + \frac{1}{2}hA_x\right]\tilde{U} + \frac{1}{2}hF_{n+1} + O(h^3)\end{aligned}$$

The Delta Form

The Delta Form

Consider the trapezoidal method again:

$$\left[I - \frac{1}{2}hA_x - \frac{1}{2}hA_y\right]U_{n+1} = \left[I + \frac{1}{2}hA_x + \frac{1}{2}hA_y\right]U_n + h(\vec{bc}) + O(h^3)$$

From both sides subtract

$$\left[I - \frac{1}{2}hA_x - \frac{1}{2}hA_y\right]U_n$$

With

$$\Delta U_n = U_{n+1} - U_n$$

We obtain

$$\left[I - \frac{1}{2}hA_x - \frac{1}{2}hA_y\right]\Delta U_n = h\Big[A_{x+y}U_n + (\vec{bc})\Big] + O(h^3)$$

The Delta Form

If we now factor the delta form of the trapezoidal method we obtain

$$\left[I - \frac{1}{2}hA_x\right]\left[I - \frac{1}{2}hA_y\right]\Delta U_n = h\Big[A_{x+y}U_n + (\vec{bc})\Big] + O(h^3)$$

Similarly the delta form of the implicit Euler method is

$$[I - hA_x - hA_y]\Delta U_n = h\Big[A_{x+y}U_n + (\vec{bc})\Big]$$

And its factored form becomes

$$[I - hA_x][I - hA_y]\Delta U_n = h\Big[A_{x+y}U_n + (\vec{bc})\Big]$$

The Representative Equation for Space-Split Operators

The Representative Equation for Space-Split Operators

Consider the representative scalar ODE for a system where $A_{x+y} = A_x + A_y$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = [\lambda_x + \lambda_y]u + a$$

In this ODE the values of λ_x and λ_y can take any combination of values of the eigenvalues of the respective one-dimensional operators

The exact solution is

$$u(t) = c\mathrm{e}^{(\lambda_x+\lambda_y)t} - \frac{a}{\lambda_x + \lambda_y}$$

Note in particular the exact steady-state solution

Example Analysis of the 2D Model Equation

Example Analysis of the 2D Model Equation

We will use the representative equation to analyze four different methods in terms of

1. stability
2. accuracy of the steady-state solution
3. convergence rate to teach steady state

The Unfactored Implicit Euler Method

$$(1 - h\,\lambda_x - h\,\lambda_y)u_{n+1} = u_n + ha$$

$$\begin{aligned} P(E) &= (1 - h\,\lambda_x - h\,\lambda_y)E - 1 \\ Q(E) &= h \end{aligned}$$

$$u_n = c\left(\frac{1}{1 - h\,\lambda_x - h\,\lambda_y}\right)^n - \frac{a}{\lambda_x + \lambda_y}$$

Unconditionally stable, exact steady-state solution, rapid convergence to steady state ($\sigma \to 0$ as $h \to \infty$)

But expensive to solve

The Factored Nondelta Form of the Implicit Euler Method

$$(1 - h\,\lambda_x)(1 - h\,\lambda_y)u_{n+1} = u_n + ha$$

$$\begin{aligned} P(E) &= (1 - h\,\lambda_x)(1 - h\,\lambda_y)E - 1 \\ Q(E) &= h \end{aligned}$$

$$u_n = c\left[\frac{1}{(1 - h\,\lambda_x)(1 - h\,\lambda_y)}\right]^n - \frac{a}{\lambda_x + \lambda_y - h\lambda_x\lambda_y}$$

Unconditionally stable, rapid convergence to steady state ($\sigma \to 0$ as $h \to \infty$), but converges to a steady-state solution that depends on h and is incorrect for large h

Less expensive to solve, but cannot be used for fast convergence (large h) to steady solutions

The Factored Delta Form of the Implicit Euler Method

$$(1 - h\,\lambda_x)(1 - h\,\lambda_y)(u_{n+1} - u_n) = h(\lambda_x u_n + \lambda_y u_n + a)$$

$$(1 - h\,\lambda_x)(1 - h\,\lambda_y)u_{n+1} = \big(1 + h^2\lambda_x\lambda_y\big)u_n + ha$$

$$u_n = c\left[\frac{1 + h^2\lambda_x\lambda_y}{(1 - h\,\lambda_x)(1 - h\,\lambda_y)}\right]^n - \frac{a}{\lambda_x + \lambda_y}$$

Unconditionally stable, exact steady-state solution, slower convergence to steady state ($\sigma \to 1$ as $h \to \infty$)

Less expensive to solve per iteration, but requires more iterations to converge to steady state

Widely used approach for steady problems

The Factored Delta Form of the Trapezoidal Method

$$\left(1 - \frac{1}{2}h\lambda_x\right)\left(1 - \frac{1}{2}h\lambda_y\right)(u_{n+1} - u_n) = h(\lambda_x u_n + \lambda_y u_n + a)$$

$$\left(1 - \frac{1}{2}h\lambda_x\right)\left(1 - \frac{1}{2}h\lambda_y\right)u_{n+1} = \left(1 + \frac{1}{2}h\lambda_x\right)\left(1 + \frac{1}{2}h\lambda_y\right)u_n + ha$$

$$u_n = c\left[\frac{\left(1 + \frac{1}{2}h\lambda_x\right)\left(1 + \frac{1}{2}h\lambda_y\right)}{\left(1 - \frac{1}{2}h\lambda_x\right)\left(1 - \frac{1}{2}h\lambda_y\right)}\right]^n - \frac{a}{\lambda_x + \lambda_y}$$

Similar properties as factored delta form of implicit Euler but second order

Possible approach for unsteady problems (2nd-order backwards more popular)